Nonautonomous and nonlinear effects in generalized classical oscillators: A boundedness theorem

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(Received 13 March 2000)

The phenomenon of exponential instability, arising in certain nonautonomous linear oscillators, motivates the question about the boundedness of amplitude and energy of the oscillators describing many physical situations. We present a rigorous result ensuring the boundedness for a class of generalized classical oscillators, characterized by symmetric potentials with only one equilibrium point. The key elements turn out to be the oscillating nature of the solutions and the presence of an autonomous part in the potential, diverging *more than quadratically* with the coordinate.

PACS number(s): 05.45.−a, 45.50.Dd, 02.30.Hq

I. INTRODUCTION

By *generalized* classical oscillator, we mean a system (not necessarily Hamiltonian), described by a single real degree of freedom $q(t)$, with first time derivative $\dot{q}(t)$, moving back and forth about a stable equilibrium position. Without loss of generality, the equilibrium position can be assumed to be the coordinate origin. Mathematically, the conditions for a system to be a generalized oscillator read as follows:

The instants $\{t_n\}$ and $\{t'_n\}$ at which $q(t_n)=0, \dot{q}(t'_n)=0$ form two unbounded sequences, such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} t'_n = \infty. \tag{1a}$$

Furthermore,

$$q(t'_n) \text{ and } q(t'_{n+1}) \text{ have opposite signs, and}$$
$$t'_n < t_n < t'_{n+1} \text{ for each } n. \tag{1b}$$

Properties (1) must be satisfied for all initial conditions except, at most, for those yielding $\lim_{t\to\infty} q(t)=0$. Generalized *Hamiltonian* oscillators with *nonautonomous* terms form a class of nontrivial problems in which the boundedness of the energy is the most important question to be addressed. In fact, the asymptotic behavior of their nonconserved energy may depend in a complicated way on the structure of the equations of motion and on the initial conditions. Typical examples are the *linear* oscillators described by the Hamiltonian

$$H_{\text{lin}} = \frac{p^2 + \Omega^2(t) q^2}{2}. \tag{2}$$

It is well known that the Hamiltonian of Eq. (2) can diverge *exponentially* in time if the linear frequency is a bounded function of time fluctuating without limit. We call this effect "exponential instability" (EI). When the fluctuation of $\Omega^2(t)$ is *periodic*, the EI is known as "parametric resonance" [1,2] and occurs if and only if the mean value of the fluctuation over a period falls into certain intervals (infinite in number), depending on the fluctuation period itself. On the other hand, a bounded but *random* fluctuation always produces EI for arbitrary amplitudes and independently of any condition of resonance (exponential localization in one dimension; see, for instance, Ref. [3]). This example motivates the interest in the asymptotic properties of generalized nonautonomous oscillators, with special reference to the boundedness properties.

If it can be done, the construction of an *invariant* function of momentum, coordinate, and time is the best possible approach to the exact integration of the equation of motion. Important results have been obtained [4,5] by expanding the invariants in series of recursively connected terms. However, the asymptotic properties of those series are in general unknown, so that the boundedness of the solutions is usually an open question. The present method does not require the use of invariants.

In Sec. II, we study *nonautonomous symmetric* Hamiltonians of the form

$$H_{\text{gen}} = \frac{p^2}{2} + \underbrace{W_t(q^2,t) + W_{\text{aut}}(q^2)}_{W(q^2,t)}, \tag{3a}$$

with equation of motion

$$\ddot{q} + \underbrace{2q\left[\partial_{q^2} W_t + \partial_{q^2} W_{\text{aut}}\right]}_{-F(q,t)} = 0, \tag{3b}$$

under the conditions

$$W_t(q^2,t); \quad \partial_{q^2} W_t(q^2,t); \quad W_{\text{aut}}(q^2);$$
$$\partial_{q^2} W_{\text{aut}}(q^2) \geq 0 \quad \forall t,q, \tag{4a}$$

$$\lim_{q^2\to\infty} \frac{W_{\text{aut}}(q^2)}{q^2} = +\infty, \tag{4b}$$

$$F(q,t)=0 \Rightarrow q=0, \tag{4c}$$

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$$W_t(0,t)\,\text{bounded.} \tag{4d}$$

Conditions (4) mean that the potential energy is nonnegative, with an *autonomous* term diverging more rapidly than q^2, and with *only one* equilibrium position in q located at $q=0$. In addition, the zero-point potential energy of the nonautonomous part must be a bounded function of the time. By means of conditions (4a) and (4c), it can be easily seen that the solutions of Eq. (3b) do actually satisfy the conditions (1) for the system to be a generalized oscillator. We prove the following theorem:

(i) *Under conditions* (4), $q(t)$, $\dot{q}(t)$, *and* $H_{\rm gen}(t)$ *are bounded*.

In proving Theorem (i), it will be seen that condition (4b) plays a special role:

(ii) *If the other conditions* (4) *are satisfied, an (autonomous, attractive) force diverging* more than linearly *with the coordinate is* sufficient *for boundedness*.

II. BOUNDEDNESS THEOREM

The main difficulty with Theorem (i) is the complicated interplay between the rapid oscillations and the smooth average trend of the solutions (what we call the "envelope") that determines the boundedness properties. Other authors have faced the same difficulty with different methods [2,6]. In particular, the boundedness has been proven by Dieckerhoff and Zehnder (DZ) in Ref. [7], via the so-called "twist theorem," for the equation of motion

$$\ddot{q}+q^{2m+1}+\sum_{j=0}^{2m}\alpha_j(t)q^j=0;\quad \alpha_j(t+\tau)=\alpha_j(t), \tag{5}$$

with time-periodic coefficients and integer powers of the coordinate. The elegant method of Ref. [7] makes use of a formalism that might look rather sophisticated to nonspecialists in dynamical systems. The present approach is more elementary and a little bit more flexible, since we can generalize the boundedness theorem to nonperiodic cases and to noninteger powers as well. However, our Hamiltonian (3a) is symmetric by definition and has only one equilibrium position, $q=0$. In this sense the case studied by DZ is more general.

Our approach is based on properties (1), characterizing the generalized oscillators. Let us introduce the so-called action-angle variables (ϕ,J), and apply the canonical transformation:

$$(q,p)\to(\phi,J);\quad q=\sqrt{2J}\cos\phi,\quad p=-\sqrt{2J}\sin\phi. \tag{6}$$

We prove the following statement:

If properties (1) *apply and* $q(t)$ *is assumed to be unbounded,*

then $\dot{q}(t)=p(t)$ *diverges with the same envelope as* $q(t)$. (7)

In fact, it is clear from Eq. (6) that $J(t)$ is strictly positive, otherwise $J(t)=0$ for each t. This is because Eq. (3b) ensures that $q(t)=0\ \ \forall t$ is a solution of the problem. In addition, $J(t)$ must contain any (eventually) diverging envelope of the squared solution and of its squared first derivative, given that $\dot{q}=p$. Therefore, the only possibility for, say, $\dot{q}^2$ to diverge in envelope more rapidly than q^2, would be

$$\lim_{t\to\infty}\cos^2\phi(t)=0\Rightarrow\lim_{t\to\infty}\sin^2\phi(t)=1\,(\text{wrong}). \tag{8}$$

From the second Eq. (8), it follows that there should exist a time $\hat{t}$ and a positive number $\delta<1$ such that

$$\sin^2\phi(t)\geqslant\delta>0\quad \forall t>\hat{t}\,(\text{wrong}). \tag{9}$$

On squaring the second Eq. (8), the strict positiveness of $J(t)$ and of $\sin^2\phi(t)$ would imply that $\dot{q}^2(t)=p^2(t)$ is in turn *strictly* positive for $t>\hat{t}$. This violates the condition (1a), which proves that the envelope of $\dot{q}(t)$ diverges *at most* as $q(t)$ for any oscillator such that $\dot{q}=p$. The same reasoning can be applied to show that $q(t)$ diverges in envelope at most as $\dot{q}(t)$. The only difference in the proof is exchanging the two trigonometric functions in Eq. (8) with each other.

For the last step of the theorem, we use the Hamilton equation for the energy itself [Eq. (3a)]:

$$\dot{H}_{\rm gen}(t)=\partial_t W_t(q^2,t), \tag{10}$$

and write Eq. (3b) in the equivalent form,

$$\dot{q}^2=2\int_{t_i}^{t}\partial_{t'}W_t dt'-2W_t(q^2,t)-2W_{\rm aut}(q^2)+2E_i, \tag{11}$$

where E_i is the energy at the initial instant t_i. In each point of the sequences $\{t_n\}$ and $\{t'_n\}$, defined by properties (1), Eq. (11) yields

$$\dot{q}^2(t_n)=2\int t_i^{t_n}\partial_t W_t(q^2,t)dt-2W_t(0,t_n)-2W_{\rm aut}(0)+2E_i, \tag{12a}$$

$$0=2\int_{t_i}^{t'_n}\partial_t W_t(q^2,t)dt-2W_t(q^2(t'_n),t'_n)-2W_{\rm aut}(q^2(t'_n))+2E_i. \tag{12b}$$

The integral on the rhs of Eq. (12b) can be expressed in terms of Eq. (12a), which yields

$$\dot{q}^2(t_n)-2\int_{t'_n}^{t_n}\partial_t W_t(q^2,t)dt+2W_t(0,t_n)+2W_{\rm aut}(0)-2W_t(q^2(t'_n),t'_n)-2W_{\rm aut}(q^2(t'_n))=0. \tag{13a}$$

On setting $\partial_t W_t=\dot{W}_t-2(\partial_{q^2}W_t)q\dot{q}$ into the integral on the lhs of Eq. (13a), then dividing by $q^2(t'_n)$, one gets

$$\frac{\dot{q}^2(t_n)}{q^2(t'_n)}+\frac{4}{q^2(t'_n)}\int_{t'_n}^{t_n}(\partial_{q^2}W_t)q\dot{q}dt-2\frac{W_{\rm aut}(q^2(t'_n))}{q^2(t'_n)}+2\frac{W_{\rm aut}(0)}{q^2(t'_n)}=0. \tag{13b}$$

By definition [property (1a)], $q^2(t_n')$ are the relative *maxima* of $q^2(t)$. Hence, if $q^2(t)$ is assumed to diverge in envelope, there exist two *unbounded* subsequences $\{\hat{t}_n'\}\subset\{t_n'\}$, $\{\hat{t}_n\}\subset\{t_n\}$, such that

$$\lim_{n\to\infty} q^2(\hat{t}_n') = +\infty(\text{wrong}), \tag{14}$$

and

$$\left[\frac{\dot{q}^2(\hat{t}_n)}{q^2(\hat{t}_n')}\right]\text{bounded}, \tag{15}$$

as implied by property (7). From the conditions (4a) and (4b) and from Eq. (14) it follows that the third term on the lhs of Eq. (13b) should diverge *negatively* and the fourth term should vanish for $n\to\infty$, when t_n' is replaced by $\hat{t}_n'$. Furthermore, Eq. (15) ensures that the first term is bounded. Therefore, the only possibility for Eq. (13a) to be satisfied asymptotically would be

$$\lim_{n\to\infty}\frac{4}{q^2(\hat{t}_n')}\int_{\hat{t}_n'}^{\hat{t}_n}(\partial_{q^2}W_t)q\dot{q}\,\mathrm{d}t = +\infty(\text{wrong}). \tag{16}$$

However, in each interval $[\hat{t}_n',\hat{t}_n]$, the function $q(t)\dot{q}(t)$ is manifestly *nonpositive*. In fact, due to property (1b), in such intervals $q(t)$ increases (decreases) from a relative *negative* minimum (*positive* maximum) to zero. Hence $q(t)$ and $\dot{q}(t)$ have necessarily opposite signs in $[\hat{t}_n',\hat{t}_n]$. Since $\partial_{q^2}W_t$ is non-negative by definition [condition (4a)], the lhs of Eq. (16) turns out to be negative. It is thus proven by reduction to the absurd that $q(t)$ and $\dot{q}(t)$ are both bounded. Condition (4d) completes the boundedness Theorem (i).

At this stage, it is useful to discuss the hypothesis of Theorem (i) on a more physical ground. The crucial points for the boundedness are the oscillating nature of the solutions and the presence of an *autonomous* term W_{aut} diverging more than quadratically with the coordinate [condition (4b)]. Should this term be eliminated, the third term on the lhs of Eq. (13b) would disappear in turn, and the next arguments would not apply anymore. Thus, the presence of W_{aut} satisfying condition (4b) is a *sufficient* condition for boundedness [statement (ii)]. In contrast to the DZ approach [7], there is no need here to assume that W_{aut} is the *most diverging* term.

III. CONCLUSIONS

The present Rapid Communication clarifies some aspects of the interplay between nonautonomous and *nonlinear* effects in generalized oscillators, with special reference to the boundedness (or stability) problem. This is especially relevant for a wide class of physical problems, whose origin is the *exponential instability* (EI) of certain *linear* oscillators [Eq. (2)]. Just to give a few examples, we mention here: the problem of the stability of beams in accelerators [8–12], the confinement of charged particles (Paul traps [13]), the squeezing effects in quantum optics [14–17], and acoustics [6]. In all these cases, Eq. (2) represents an approximation, neglecting higher-order nonlinear effects, that are unavoidable in practical applications. As far as Eq. (2) is concerned, we stress that the *randomly* fluctuating frequency is a particularly intriguing case. On one hand, it seems to outline the solution of a long-standing technological problem, i.e., using the environment as an infinite source of randomly fluctuating energy, to be extracted and concentrated in a controllable system. On the other hand, the same effect will sound alarming, in some circumstances. For example, the energy of the transversal modes of an elastic bar, under the action of longitudinal time-varying forces ([2], Chapter 17) is just determined by Eq. (2). In this case the fluctuating part of $\Omega^2(t)$ is proportional to the longitudinal force. As a structural part of a vehicle or a building, a bar can be influenced by random-varying forces, even in quite normal conditions. The consequences should be always (or could have been, sometimes) catastrophic, if some contrasting mechanisms did not suppress the EI. Statement (ii) addressed the nonlinearity as an important mechanism contrasting the EI [18]. The present communication should stimulate a *quantitative analysis* on the ability of the nonlinear effects to suppress the EI, under realistic (though extreme) physical conditions. To our knowledge, this problem was presumably addressed long ago [19] for the structural stability of aircraft, in the case of *parametric resonance* (periodical fluctuations). It is not clear, however, if the same question has been considered in the case of *random* fluctuations too, in more recent times. If this was *not* the case, the present communication should sound as an alarm bell for engineers and technologists.

In a forthcoming paper, we will go beyond the boundedness theorem proven above, by studying the classical Hamiltonian

$$H=\frac{p^2+\Omega^2(t)q^2}{2}+\frac{\alpha}{\gamma}|q|^{\gamma};\quad \gamma>2, \tag{17}$$

as a special case of Eq. (3a), satisfying conditions (4). The interplay between EI and nonlinearity is expressed by Eq. (17) in the simplest possible form. Indeed, if the quadratic part is exponentially unstable, the low-energy dynamics resulting from the nonlinear terms exhibits a very rich structure, including a *critical transition*, driven by the initial energy of the generalized oscillator.

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